

**Definition 1. (Locus of an Equation)**

The *cartesian plane* is the set of all ordered pairs of real numbers. These ordered pairs are the *points* in the plane.

Consider an equation in two variables,  $x$  and  $y$ . The *locus* of the equation is the set of points  $(x, y)$  in the cartesian plane which, when plugged into the equation, make it true.

The *equation of a line* is an equation in  $x$  and  $y$  such that a point is on the line if and only if it satisfies the equation.

**Example 1. (Line Equation as a Test)**

Consider the line  $y = 2x - 5$ . Determine if either of the points  $(1, 7)$  or  $(4, 3)$  are on the line.

*Solution.* View the equation as a test, which checks to see if a point is on the line. We plug the points into the equation to see if they make it true.

(a)  $(1, 7)$  Since  $7 = 2(1) - 5$  is false, the point IS NOT ON the line.

(b)  $(4, 3)$  Since  $4 = 2(3) - 5$  is true, the point IS ON the line.

□

The *equation of a circle* is an equation in  $x$  and  $y$  such that a point is on the circle if and only if it satisfies the equation. Consider a circle whose center is  $(h, k)$  and whose radius is  $r$ . If  $(x, y)$  is an arbitrary point on the circle, then the distance from  $(x, y)$  to  $(h, k)$  is  $r$ . The distance formula, which comes from the Pythagorean Theorem, states this as

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

Squaring both sides leads to the standard form of the equation of a circle.

**Definition 2. (Equation of a Circle)** The equation of a circle with center  $(h, k)$  and radius  $r$  is

$$(x-h)^2 + (y-k)^2 = r^2.$$

**Example 2. (Circle Equation as a Test)** Consider the equation  $(x-1)^2 + (y+2)^2 = 25$ . Find its center and radius. Is either of the points  $(3, 4)$  or  $(5, 1)$  on the line?

*Solution.* Match the given equation to  $(x-h)^2 + (y-k)^2 = r^2$ . We see that  $h = 1$ ,  $k = -2$ , and  $r^2 = 25$  or  $r = 5$  (the radius must be positive). Thus, the center is  $(1, -2)$  and the radius is  $r$ .

Now use the equation as a test to check if a given point is on the circle. Plug the points into the equation to see if they make it true.

(a)  $(3, 4)$  Since  $(3-1)^2 + (4+2)^2 = 2^2 + 6^2 = 4 + 36 = 40 \neq 25$ , the point IS NOT ON the circle.

(b)  $(5, 1)$  Since  $(5-1)^2 + (1+2)^2 = 4^2 + 3^2 = 16 + 9 = 25$ , the point IS ON the circle.

□

**Example 3. (Find the Center and Radius)**

The locus of the equation  $x^2 - 8x + y^2 + 14y = 12$  is a circle. Find its center and radius.

*Solution.* We complete the square. Add 16 and 49 to both sides, thusly:

$$x^2 - 8x + 16 + y^2 + 14y + 49 = 12 + 16 + 49.$$

Associate the appropriate terms, and add the right hand side:

$$(x^2 - 8x + 16) + (y^2 + 14y + 49) = 77.$$

Factor to get:

$$(x-4)^2 + (y+7)^2 = 77.$$

The center is  $(4, -7)$  and the radius is  $\sqrt{77}$ .

□

**Example 4. (Find the Intersection of a Line and a Circle)**

Find the points of intersection of the circle  $(x - 3)^2 + (y - 4)^2 = 72$  and the line  $y = x + 1$ .

*Solution.* Substitute the equation of the line into the equation of the circle, and solve to  $x$ . We get:

$$\begin{aligned}(x - 3)^2 + ((x + 1) - 4)^2 &= 72 \Rightarrow (x - 3)^2 + (x - 3)^2 = 36 \\ \Rightarrow (x - 3)^2 &= 36 \\ \Rightarrow x - 3 &= \pm 6 \\ \Rightarrow x &= 9 \text{ or } x = -3.\end{aligned}$$

Plug this into the line to get the corresponding  $y$ : if  $x = 9$ ,  $y = x + 1 = 10$ , and if  $x = -3$ ,  $y = x + 1 = -2$ . So, the two points of intersection are

$(9, 10) \text{ and } (-3, -2).$

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